Fractal dimension and non-linear dynamical processes

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ABSTRACT

Mandelbrot, Falconer and others have demonstrated the existence of dimensionally invariant geometrical properties of non-linear dynamical processes known as fractals. Barnsley defines fractal geometry as an extension of classical geometry. Such an extension, however, is not mathematically trivial! Of specific interest to those engaged in signal processing is the potential use of fractal geometry to facilitate the analysis of non-linear signal processes often referred to as non-linear time series.

Fractal geometry has been used in the modeling of non-linear time series represented by radar signals in the presence of ground clutter or interference generated by spatially distributed reflections around the target or a radar system. It was recognized by Mandelbrot that the fractal geometries represented by man-made objects had different dimensions than the geometries of the familiar objects that abound in nature such as leaves, clouds, ferns, trees, etc.

The invariant dimensional property of non-linear processes suggests that in the case of acoustic signals (active or passive) generated within a dispersive medium such as the ocean environment, there exists much rich structure that will aid in the detection and classification of various objects, man-made or natural, within the medium.

1. INTRODUCTION

The processing of non-linear dynamical signal processes is a relatively new approach to signal process modeling and analysis as is evidenced by the sparseness of the literature in such prestigious journals as the IEEE's Acoustics, Speech and Signal Processing. While just released, the 1992 Proceedings Volumes 1–5 of the International Conference on Speech and Signal Processing held in San Francisco, CA in November 1991 for the first time features some papers on non-linear dynamical physical processes, and fractal processes that suggest there is an increasing interest in the modeling and analysis of non-linear processes as represented by acoustic signal processes in a dispersive medium.

The analysis of non-linear processes has not been feasible prior to the development of high speed digital computers. Even so, most investigators concerned with the analysis of non-linear processes preferred to simplify and linearize such processes whenever possible and apply suitable approximation techniques from which to draw whatever inferences and conclusions that could be squeezed out of such an approach.

While the fractal modeling and analysis of non-linear dynamical processes has not received a great deal of attention from the signal processing community at the present time, Marteau and Abarbanel[1] have been studying acoustic FM signal processes emersed in fractal noise. The primary objective of their studies was the separation of a signal process from fractal noise. Constantikes[2] and Butterfield[3] have conducted independent studies in the detection of radar targets emersed in extensive ground clutter.

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The representation of non-linear signal processes as fractal processes leads to the separation of a composite signal process in terms of the calculated fractal dimensions of a signal and a noise process.

It is generally agreed that fractal dimensions of noise processes are different from those processes representing the reflection of acoustic and/or e.m. signals from such targets as ships or airplanes. Of course, the estimated measures of fractal dimensions are not necessarily those generated by simple scatterers existing within the physical realm of some non-linear dynamical process.

Fractals are represented by irregular sets whose geometry exhibits invariant dimensions D as fractions or rational numbers. Mandelbrot[4] defines fractals as self-similar sets, whose spatial characteristics remain qualitatively similar under expansion or contraction. Chatterjee and Yilmaz[5] note in their recent work that fractal processes are a descriptive method of analyzing irregular sets which enable a mathematical description of a process resulting from the behavior of a non-linear dynamical system. Such behavior has been popularly referred to in the literature as chaos.

Devaney[6] states that chaos is a much abused term in dynamical systems. But since there is no generally accepted definition of chaos, Devaney suggests that we let $\mathcal{F}: M \rightarrow M$ be a map where M is a metric space, $\mathcal{F}$ is then chaotic if (1) $\mathcal{F}$ is sensitive to initial conditions, (2) $\mathcal{F}$ is topologically transitive, (3) periodic points of $\mathcal{F}$ are dense in M. The existence of periodic points imply that there is an element of regularity on the mapping.

We shall illustrate the generation of some fractal geometries by the representation of a noisy signal process by a Weierstrass function and a random Von Koch curve.

Of particular importance to signal processing applications are measures of fractal dimensionality. Several measures of fractal dimensionality have been proposed but most of these measures are difficult to estimate from models of non-linear dynamical processes, let alone simple maps thereof. A measure of fractal dimensionality D, that has excellent properties is the so called Hausdorff-Besicovitch dimension D_H. One version suggested as an upper bound for this measure appears in a recent expository paper by Chatterjee and Yilmaz. Viewing an attractor as a subset on R^n, the attractor is to be covered with volume elements of size ε. These elements can represent spheres with diameters ε or cubes with sides ε. Letting $N(\varepsilon)$ be the smallest number of cubes or spheres it takes to cover the attractor, then $N(\varepsilon) = c\varepsilon^{-n}$ for some constant $c > 0$. This yields an upper bound for the Hausdorff-Besicovitch dimension

$$D_H \leq D = \lim_{\varepsilon \to 0} \frac{\log(\varepsilon)}{\log(1/\varepsilon)}$$

if this limit exists. We know that, is not necessarily an integer. In fact $D = 1$ represents a line or curve segment and $D = 2$ represents a bounaeu surface or any manifold in R^2. The simplest fractal set that one can generate is illustrated nicely by the so called middle third Canton set f as demonstrated in Falconer[7]. If $D = \text{Log } 2 / \text{Log } 3 = 0.6309...$ as calculated by the limit relation above, then the (Hausdorff-Besicovitch) dimension $D_H = D$.

There are many estimates of the dimensionality of a fractal process such as the Box dimension, and correlation dimension, but, in general, D is a measure of the density of a given set within its metric space of residence.

It would seem reasonable then to expect that a measure of dimensionality would be consistent and sufficiently robust to the extent that it would permit the classification of a non-linear physical process as generated by man or nature. Thus, fractal dimensionality should enable us to separate composite signal processes into their representative natural and man-made components. Unfortunately, fractal dimensionality in and of itself does not appear to be sufficient to the obtainment of a unique classification of a given process with respect to other non-linear processes of a similar nature. We, therefore, must discover additional measures or metrics that will support the unique classification of any particular non-linear process if such metrics exist.

2. FRACTAL DIMENSIONS

Michael Barnsley[8], using the basic definition of the limit expression for D as discussed briefly in Section 1, presents some very fine examples of using fractal dimension in the analysis of experimental data as well as calculating the fractal dimension of the Cantor set. Barnsley has also discussed in very understandable detail the Hausdorff-Besicovitch Fractal Dimension, D_H. Of particular importance, this dimension can be used in comparing the sizes of sets having the same fractal dimension. Unfortunately, the utility of this fractal dimension is not straightforward in the estimation of fractal dimensions of sets representing physical or experimental data. Since experimental data sets representing certain behavioral aspects of non-linear dynamic processes can have the same fractal dimension, one must be very careful as to the conclusions he or she draws from the comparison of the dimensionality alone. It is to be hoped that as further developments continue to take place in the field of fractal geometry and non-linear dynamics, additional results will be obtained with which to support the tasks of process classification.

2.1 Hausdorff-Besicovitch dimension

This definition of fractal dimension would not be easily calculated from experimental data. However, it would be useful in comparing the sizes of sets seemingly having the same fractal dimension. This event could occur easily within the realm of acceptable experimental errors during the data collection process. One important property of this dimension is that it is translationally and rotationally invariant. Thus, if any two sets are regarded as the same, there would exist a bi-Lipschitz mapping between them. First we see that if $f{:}F \rightarrow R^m$ is a mapping for a continuous f and such that $|f(x) - f(y)| \leq c\,|x - y|\,(x,y \,\varepsilon\, F)$, then f is a Lipschitz map. Therefore, the Hausdorff dimension of $f(F) \leq$ the Hausdorff dimension of F, and as Falconer also shows, if f is a bi-Lipschitz transformation, i.e., $c_1\,|x - y| \leq |f(x) - f(y)| \leq c_2\,|x - y|$ for $0 < c_1 \leq c_2 < \infty$ then the Hausdorff dimension of $f(F)$ = the Hausdorff dimension of F. While all of these results are necessary for a comparison of fractal sets, we must keep in mind that the dimensions of a fractal set do not alone provide us with sufficient information to conclude that two such sets represent the same non-linear dynamical behavior.

To illustrate these definitions of fractal dimension and Hausdorff-Besicovitch dimension, we shall examine their application to the calculation of the dimension of the middle third Cantor set. In this case the Hausdorff-Besicovitch and the box-like fractal dimension calculations yield the same result even though they are not similarly defined. Therefore, we can conclude at this point that fractal dimensional metrics are consistent with reasonable robustness as relates to the effect of estimation errors.

Falconer uses a very realistic heuristic calculation for the dimension of the middle third Cantor set, shown in Figure 1, generated by the requisite iteration of self-similar processes. First, defining the Hausdorff dimension rigorously we have, given a set F, the dimension of F is given by the following relationships, $\dim F = \inf\{s{:}H^s(F) = 0\} = \sup\{s{:}H^s(F) = \infty\}$, where inf is the greatest lower bound and the sup is the least upper bound shown graphically in Figure 2. Thus, the Hausdorff dimension occurs for that value s at which the jump from ∞ to 0 occurs. If $s = \dim F$, then H^S may be 0, infinite or such that $0 < H^S(F) < \infty$. Then returning to the middle third Cantor set, the reasoning was as follows: The set F splits into a right and left part with $F_r = F \cap [2/3, 1]$, i.e., both sets are geometrically similar but scaled by 1/3 with $F = F_L \cup F_R$, i.e., F is the union of disjoint sets (F_L, F_R). Assume $\dim F = s$. Then for any s,

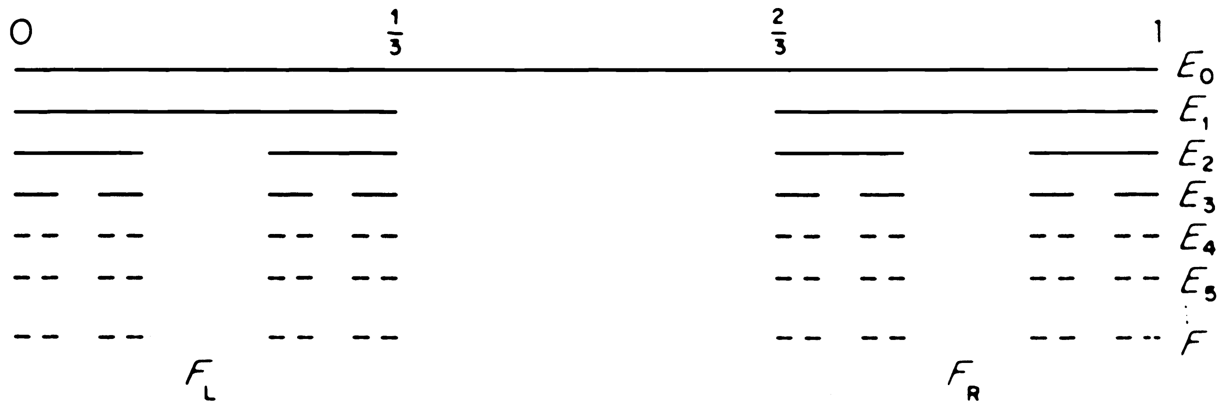

Figure 1 Construction of the middle third Cantor set F, by repeated removal of the middle third of intervals. Note that F_L and F_R, the left and right parts of F, are copies of F scaled by a factor $\frac{1}{3}$. *Presented here with the permission of John Wiley & Sons, Ltd.*

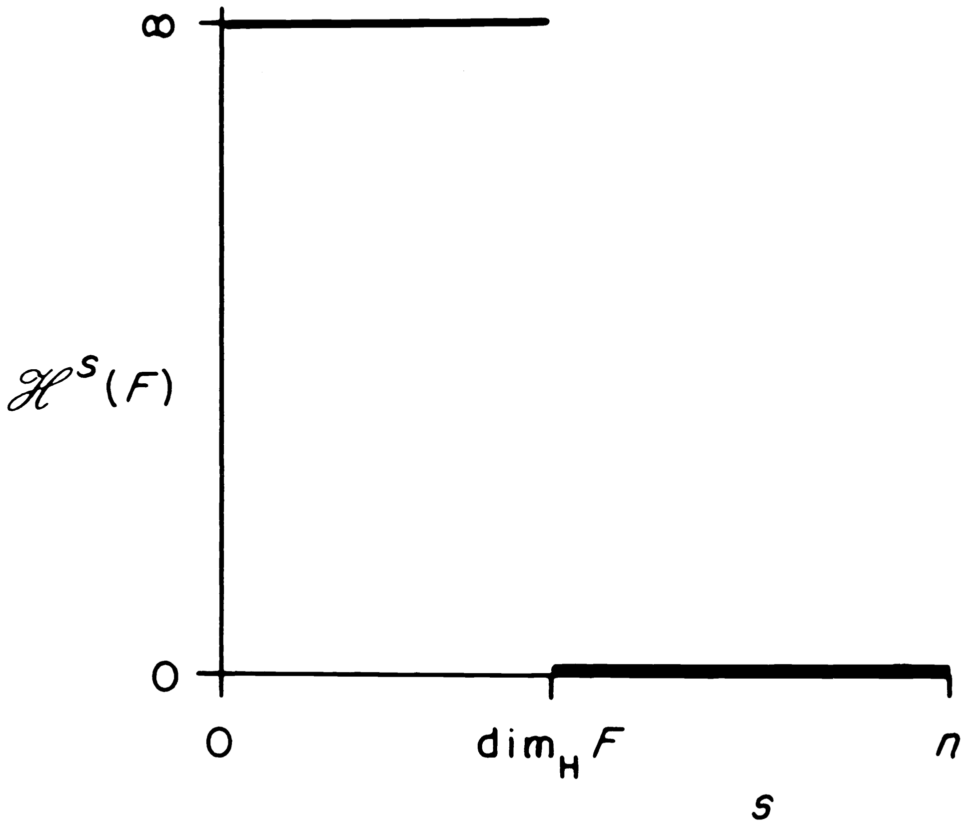

Figure 2 Graph of $\mathcal{H}^s(F)$ against s for a set F. The Hausdorff dimension is the value of s at which the 'jump' from ∞ to 0 occurs. *Presented here with the permission of John Wiley & Sons, Ltd.*

$H^S(F) = H^S(F_L) + H^S(F_R) \; \frac{1}{3}^s \; H^S(F) + \frac{1}{3}^s \; H^S(F)$, then let the critical value $s = \dim F$ and then $0 < H^S(F) < \infty$. Now divide $H^S(F) = \frac{1}{3}^s H^S(F) + \frac{1}{3}^s H^S(F)$ by $H^S(F)$ and this yields

$1 = 2\,(1/3)^s$ which on taking log both sides become

$0 = \log 2 - s \log 3$ $\;s = \log 2 / \log 3 = 0.6509...$, the fractal dimension of the middle third Cantor set. Now let us do the calculation for the dimension of the same set using a box counting technique as demonstrated by Barnsley. Using 2^n boxes of side length $(1/3^n)$, we conclude that $D(F_c) = \lim_{n\to o} \left(\frac{\log(2^n)}{\log(3^n)}\right) = \frac{\log 2}{\log 3} = 0.6509....$ Now if we had two different fractal sets with the same dimension D, we could conclude that the sets were the same since the Hausdorff-Besicovitch dimension is the same; hence, the sets are the same "size." However, if we are truly concerned about two fractal sets being the same in more specific ways, such as approaching a unique classification, it is apparent that we need to consider additional attributes beyond those of fractal dimensions alone.

2.2 Weierstrass functions

Weierstrass functions are examples of sets of continuous functions which are nowhere differentiable. While previously proved by Hardy, circa 1916, Hewitt and Stromberg[9] proved a theorem similar to that of Hardy using the fundamental definition of the derivative which demonstrates that the slopes of the successive strokes between points of continuity are infinite (see Figure 4).

We consider these functions because they would model noisy signal processes which could be used for example to hide coded signals more completely than could be accomplished in the case of spread spectrum signals generated by pseudo-random codes, etc.

A Weierstrass function presented and discussed by Falconer is

$$f(t) = \sum_{k=1}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t)$$

for $\lambda > 1$ and $1 < s < 2$. For λ large, the box dimension, shown in Figure 3, of the graph f equals s. The following graphs, Figures 4 and 5, of $f(t)$ for $\lambda = 1.5$ and $s = 1.1, 1.3, 1.5$, and 1.7 are represented here with the permission of John Wiley and Sons, as they appear in Falconer. It is further stated that the Hausdorff-Besicovitch dimension is at most s and some investigators conjecture that it equals s for most values of λ. It is said to be conjecture since no rigorous proof of equality has yet been forthcoming.

2.3 Von Koch curve

Also illustrated here in Figure 6, with the permission of John Wiley and Sons, is a fractal curve generated from a unit length line segment. The construction of this curve is somewhat similar to that of construction of the middle third Cantor set. The construction begins with a unit line segment E_0. Then sets E_1 through E_k are generated by the process of removing the middle third of E_0, leaving two segments representing the set E_1. The removed middle third is used to form the sides of an equilateral triangle. The set E_2 is constructed similarly (as per the illustration). Thus, E_k is constructed from the middle third of each straight line segment of E_{k-1}. The Von Koch curve has the fractal dimension $\log 4 / \log 3 \sim 1.26$; hence it has infinite length but zero area. Falconer points out that a curve made up of m copies of itself and scaled by a factor r, has a similarity dimension $-\log m / \log r$. In the case of the Von Koch curve we observe that $m = 4$ and $r = 1/3$.

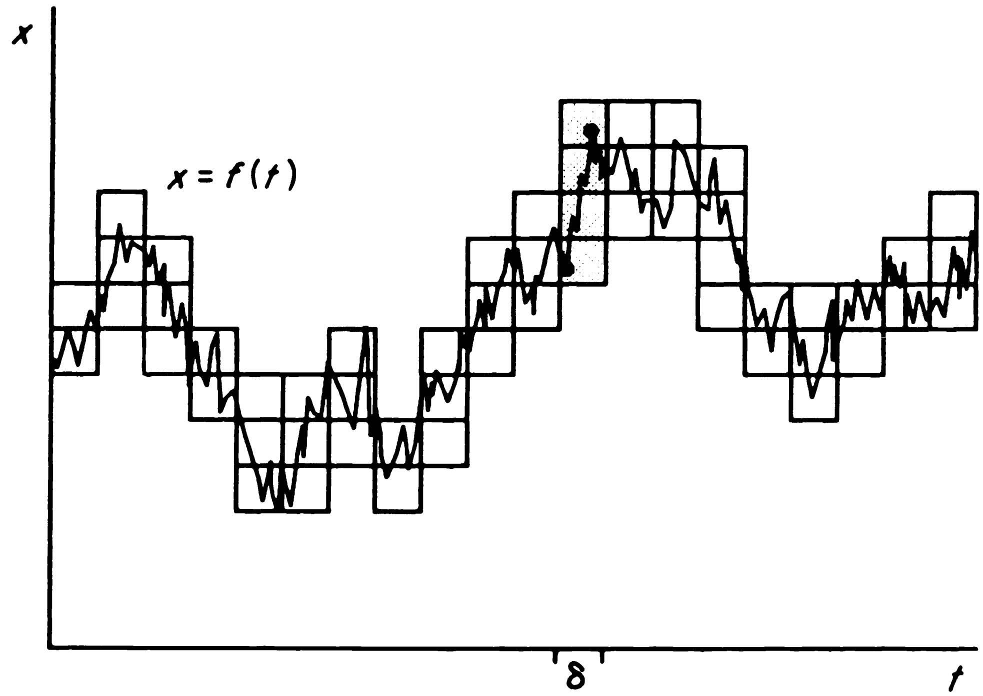

Figure 3 The number of δ-mesh squares in a column above an interval of width δ that intersect graph f is approximately the range of f over that interval divided by δ. Summing these numbers gives estimates for the box dimension of graph f. *Presented here with the permission of John Wiley & Sons, Ltd.*

2.4 Analysis of experiments

The fractal analysis of experimental data will require significant knowledge as to the nature of the physical process generating any set of characteristic observations being collected. Caution must be exercised as to the effects of the measurement system on the data collection process. As an example, it is well known in the realm of signal processing procedures that filters tend to smooth and linearize the sample being collected. By so doing, much of the valuable non-linear structural "information," which would provide valuable information as to the nature of the underlying process, is lost. Barnsley discusses the problems associated with the collection and interpretation of "real world" experimental data. In particular, he has focused our attention on some of the problems associated with a unique measure of the fractal dimensionality of experimental data sets. In this regard, Barnsley examined an experiment conducted by Strahle[10], which compares two sets of experimental data that were obtained from the same experiment using different measurement techniques. Strahle obtained the same fractal dimensionality for the two data sets and concluded from this that the two data sets have a common source which is physical chaos. In one case the data set consisted of the measurements of the scattering of a laser beam by a turbulent jet flame shown in Figure 7. The other set of experimental measurements, shown in Figûres 8 and 9, were obtained as measurements of the voltage across a wire in the flame. As shown in Figure 10, it was concluded by the use of a box counting technique over the realm of the two different appearing graphs of the experimental data that both have dimensionality $D \approx 1.5$ over range scales 64×10^{-5} to 0.32768 sec. This further suggests that the common source is the dynamics of chaos resident in the jet flame exhaust, and, therefore, fractal dimensions can be used to quantify chaotic behavior.

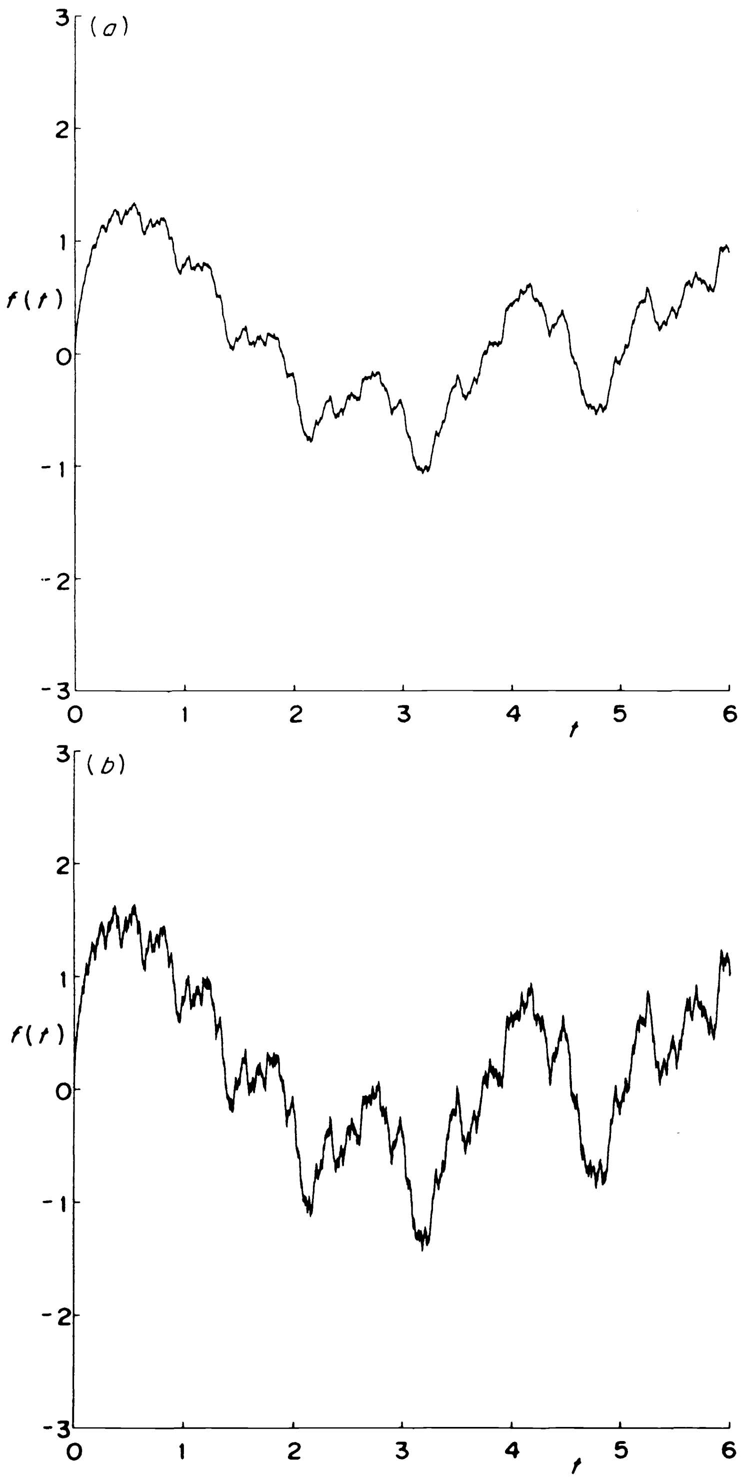

Figure 4 The Weierstrass function $f(t) = \sum_{k=0}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t)$ *with* $\lambda = 1.5$ and (*a*) $s = 1.1$, (b) $s = 1.3$, (c) $s = 1.5$, (d) $s = 1.7$. *Presented here with the permission of John Wiley & Sons, Ltd.*

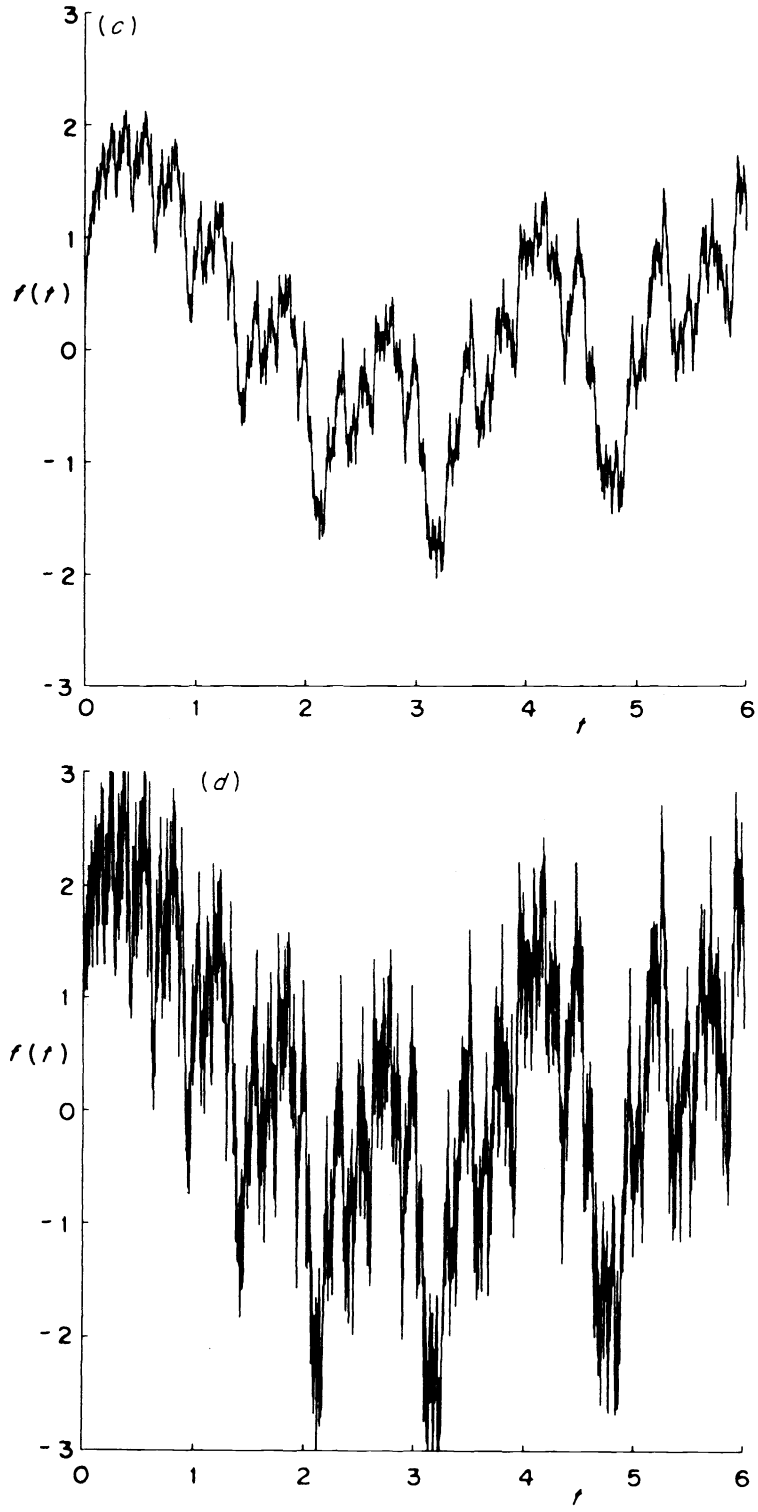

Figure 5 The Weierstrass function $f(t) = \sum_{k=0}^{\infty} \lambda^{(s-2)k} \sin(\lambda^k t)$ with $\lambda = 1.5$ and (*a*) $s = 1.1$, (b) $s = 1.3$, (c) $s = 1.5$, (d) $s = 1.7$. ***Presented here with the permission of John Wiley & Sons, Ltd.***

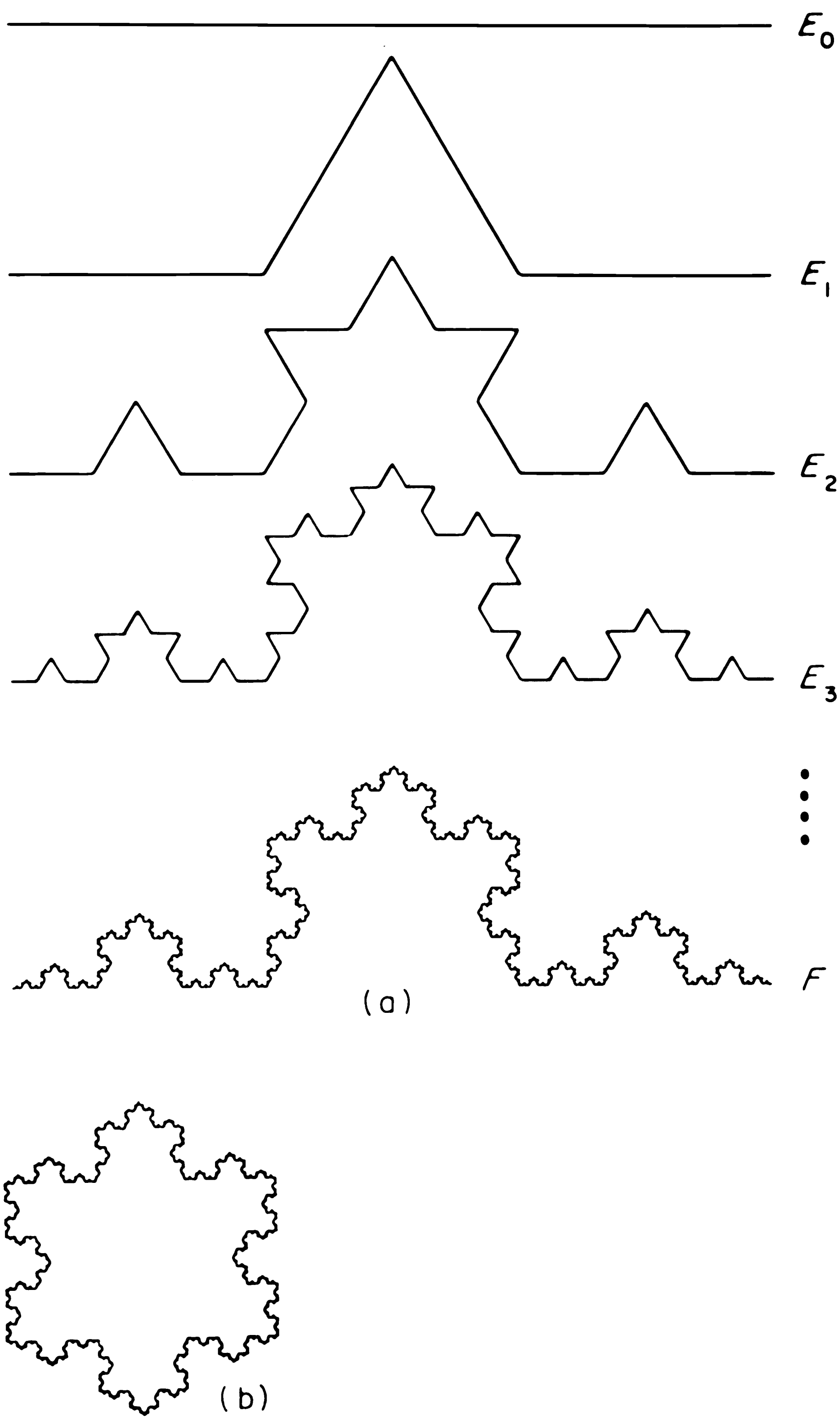

Figure 6 (a) Construction of the Von Koch curve F. At each stage, the middle third of each interval is replaced by the other two sides of an equilateral triangle. (b) Three Von Koch curves fitted together to form a snowflake curve. *Presented here with the permission of John Wiley & Sons, Ltd.*

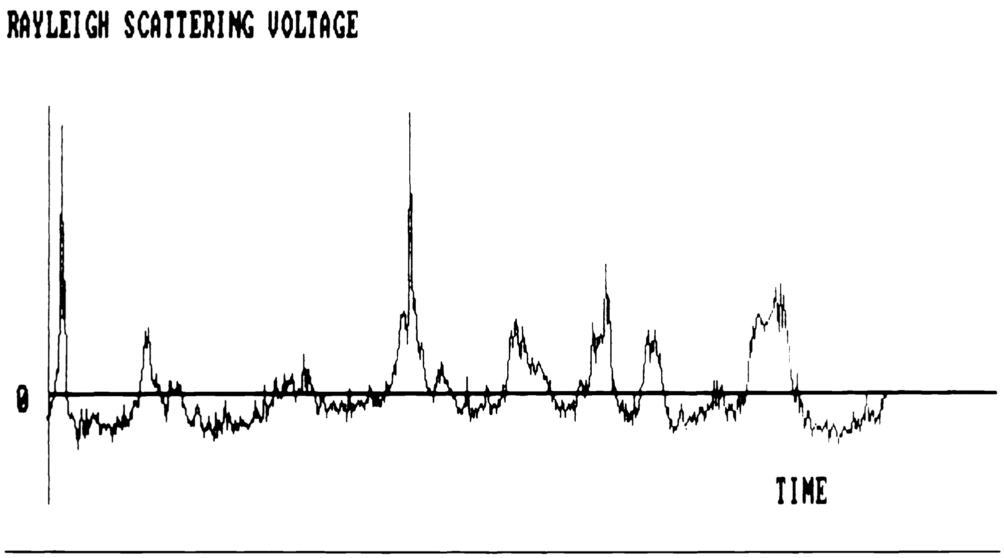

Figure 7 Graph of voltage as a function of time from an experimental probe of a turbulent jet. In this case, the probe measures scattering of a laser beam by the flame. *By Michael Barnsley from* *__Fractals Everywhere__* *by Michael Barnsley, Copyright 1988 by Academic Press, Inc.*

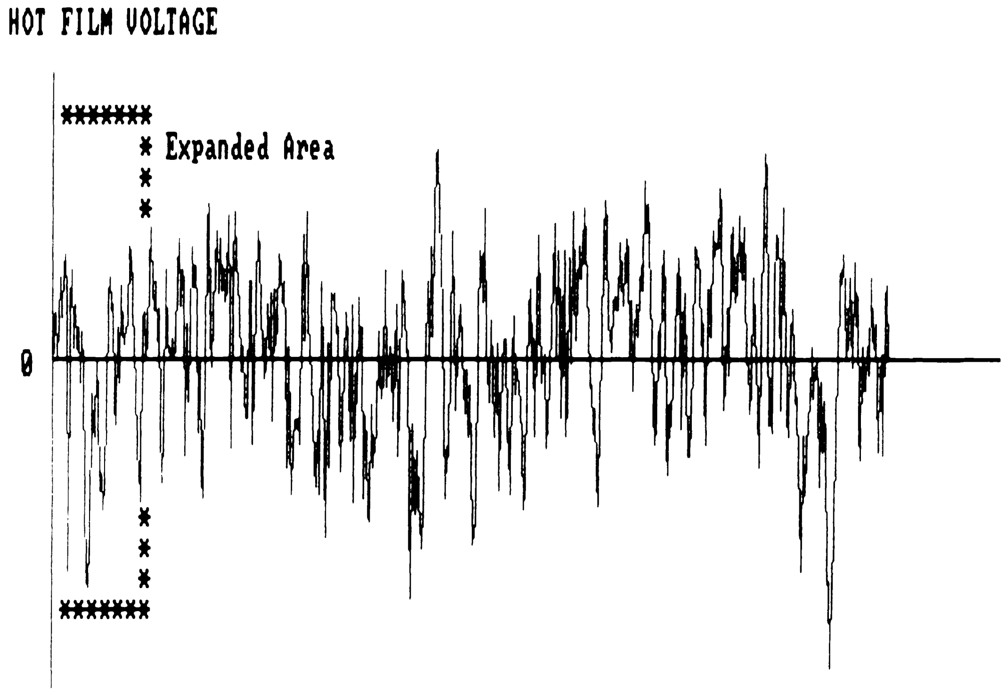

Figure 8 Graph of voltage as a function of time from an experimental probe of a turbulent jet. In this case, the probe measures the voltage across a wire in the flame. This data has a definite fractal character, as demonstrated by the expanded piece shown in Figure 9. *By Michael Barnsley from* *__Fractals Everywhere__* *by Michael Barnsley, Copyright 1988 by Academic Press, Inc.*

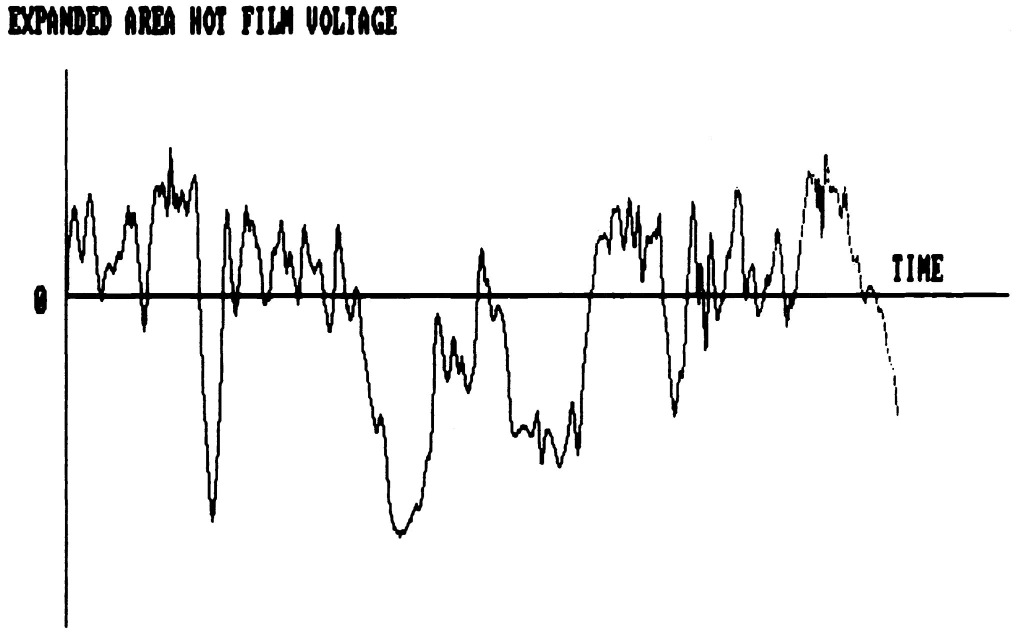

Figure 9 A blow-up of a piece of the graph in Figure 8. *By Michael Barnsley from Fractals Everywhere by Michael Barnsley, Copyright 1988 by Academic Press, Inc.*

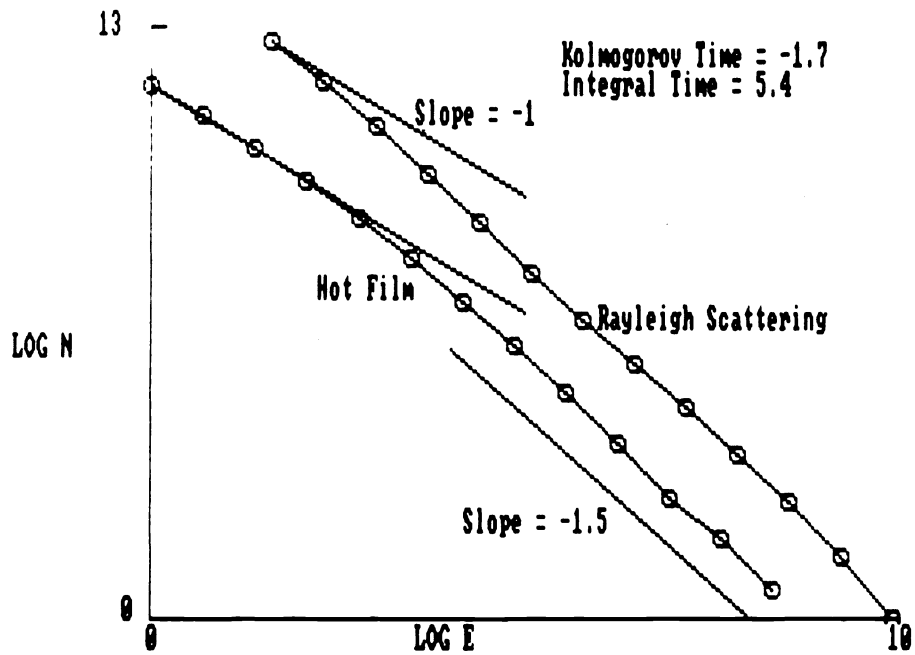

Figure 10 Graphical analysis of Box Counts associated with experiments (*a*) and (*b*). The data analyzed is illustrated in Figures 7 and 8. The two data streams, which come from probes of a single turbulent system, when analyzed in exactly the same way, yield the same value $D = 1.5$ for the fractal dimension. This suggests that, despite the different appearances of their graphs, there is a common source for the data. This source is chaotic dynamics of a certain special flavor and character. The fractal dimension provides a measurable symptom of the brand of chaos. *By Michael Barnsley from Fractals Everywhere by Michael Barnsley, Copyright 1988 by Academic Press, Inc.*

Not only is the physical nature of the sample process important, but, as we previously mentioned, great care must be exercised in the gathering of the sets of experimental data to ensure that alterations of these observations do not occur in such a way that the properties of self-similarity, non-linear structure, etc., are lost. In the signal processing of electromagnetic signals such as radar pulses, it is assumed that such processes, when not immersed in ground clutter (between the observer) and the target are generally linear. Hence, there would be very little to be gained by analyzing a linear process as though it were a self-similar, non-linear process since the fractal dimensionality of such a process would not exist. Therefore, depending upon the nature of the sample space, it would be reasonable to expect that linear behavior would result in a fractal dimension calculation of unity, two or three, the usual Euclidean dimension of a line, plane, or three-dimensional space. It is clear that much more research must be undertaken for the express purpose of finding additional measures of fractal behavior which would aid and abet a unique classification of physical processes in terms of certain observables in addition to fractal dimensionality.

3. CONCLUSIONS

(a) The unique modeling and analysis of experimental data even known to be representative of a given dynamical non-linear process will require physically descriptive behavioral measures in addition to fractal dimensionality.

(b) Experimental measurements of non-linear dynamical behavior must retain the non-linearity characteristics of the process or the fractal nature of the process will not otherwise be retained.

(c) Hausdorff-Besicovitch dimension will provide a measure of size difference of two otherwise similar sets in terms of their fractal dimension D_H.

(d) Additional research is required to ascertain the existence of metrics other than fractal dimensionality which would provide additional measures of so called chaos existing within the realm of physically generated non-linear dynamical processes. Acoustic signals propagating in a non-linear dispersive medium (typical ocean environment), turbulent flow phenomena generated by the motion of man-made bodies in a ocean environment, turbulent behavior exhibited by compressible (aerodynamics) and incompressible (hydrodynamics) fluid flow are classical examples of physically generated non-linear dynamical processes.

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